Meet #1
October 2006

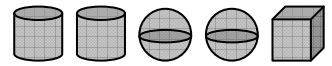
Intermediate Mathematics League of Eastern Massachusetts

Average team score: 106.8

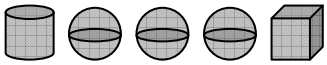
Meet #1
October 2006

Category 1
Mystery
Meet #1, October 2006

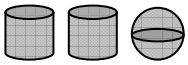
1. In the picture below, same objects have the same weight and different objects have different weights. Scales A, B, and C give the total weight in pounds for the objects above them. How many pounds does one cube weigh?



A: 67 lbs.



B: 68 lbs.

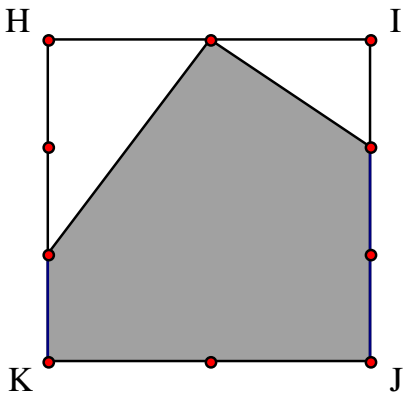


C: 43 lbs.

2. If the pattern of letters below continues, what will the 100th letter be?

MATHFUNMATHMATHFUNFUNMATHMATHMATHFUNFUNFUN...

3. Quadrilateral HIJK, shown below, is a square. Sides HI and JK have been bisected, and sides HK and IJ have been trisected. What portion of the area of square HIJK is shaded? Express your answer as a fraction in lowest terms.



Answers

1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #1, October 2006

Average score: 1.1 answers correct

Answers

1. 9

2. F

3. $\frac{3}{4}$

1. The only difference between scales A and B is that one cylinder has been swapped for a sphere. Since the weight went up by 1 pound, we know that a sphere weighs one pound more than a cylinder. Taking this information to scale C, we can imagine swapping the sphere for a cylinder. The weight would go down by one pound, so we would have three cylinders weighing 42 pounds. The cylinder must weigh 14 pounds and the sphere must weigh 15 pounds. Taking these weights to scale A, we see that we have 14 + 14 + 15 + 15 = 58 pounds in cylinders and spheres, so the cube must weigh 67 – 58 = **9** pounds.

~~Some Incorrect Answers Seen~~

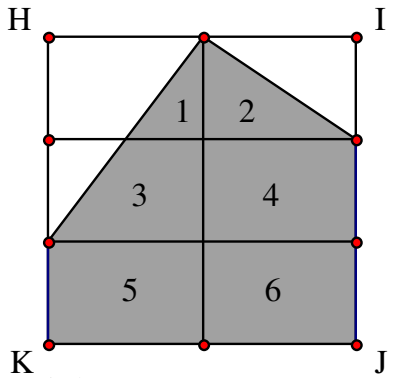
1. 15, 18

2. A

3. ½

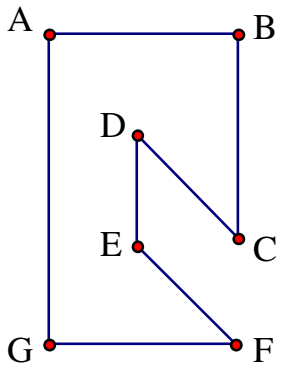
2. There are 7 letters in the original MATHFUN. There are 14 letters when the words are doubled, 21 letters when the words are tripled, and so on. We should add multiples of seven until we get close to 100: 7 + 14 + 21 + 28 + 35 = 105. The 100th letter will be in the midst of a string of five FUN's. If we remove the final UNFUN, we can see that the 100th letter must be an **F**.

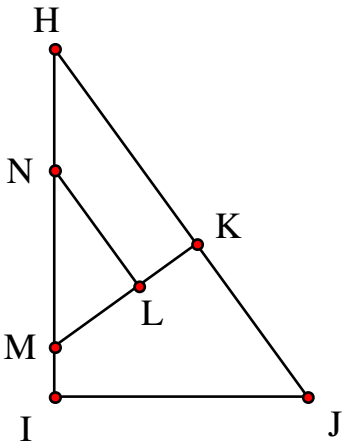
3. Regions 4, 5, and 6, in the figure below, are each one sixth of the square. Regions 1 and 3 together make another sixth. Region 2 is one half of a sixth. In all, we have $\frac{4\frac{1}{2}}{6}$, which simplifies to $\frac{9}{12}$ and then to $\mathbf{\frac{3}{4}}$.



Category 2
Geometry
Meet #1, October 2006

1. In heptagon ABCDEFG, drawn accurately at right, all angles are multiples of 45 degrees. How many degrees are in the sum of interior angles D and E?





2. In the figure at left, angles HIJ, HKM, and MLN are right angles. If the measure of angle MNL is 36 degrees, how many degrees are in the measure of angle HJI?

3. If you subtract twice an angle from its supplement, you get half its complement. How many degrees are in the measure of this angle?

Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #1, October 2006

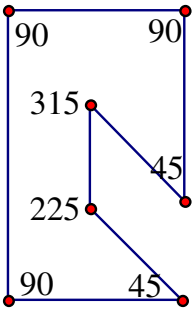
Average score: 1.3 answers correct

Answers

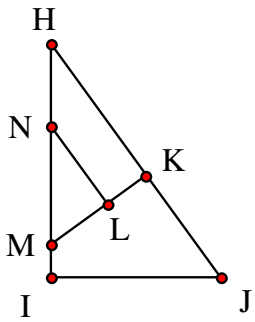
1. 540

2. 54

3. 54



1. The angle measures are given here. The desired sum is 315 + 225 = **540**.



2. Triangles HIJ, HKM, and NLM are similar triangles. This means they have the same angle measures. We know that angle MLN is 90 degrees and angle MNL is 36 degrees, so angle LMN must be 180 – 90 – 36 = 54 degrees. Angle HJI is also **54** degrees.

3. If we call our unknown angle x, then twice the angle is $2x$, the supplement is $180 - x$ and the complement is $90 - x$. Translating the sentence to algebra, we get the following equation:

$$(180 - x) - 2x = \frac{90 - x}{2}$$

Simplifying the left side, we get

$$180 - 3x = \frac{90 - x}{2}$$

Doubling both sides, we get

$$360 - 6x = 90 - x$$

Adding $6x$ to both sides of the equation, we get

$$360 = 90 + 5x$$

Subtracting 90 from both sides, we get

$$270 = 5x$$

Finally, dividing both sides by 5, we find that $54 = x$. So the measure of the unknown angle is **54** degrees.

Some Incorrect Answers Seen

1. 180

2.

3. 0, 60, 90

Category 3
Number Theory
Meet #1, October 2006

1. What is the least three-digit number that is divisible by exactly three different prime numbers?

2. The five-digit number 35*N*2*N* is divisible by 2, 3, 4, 6, 8, 9, and 12, among other numbers. What is the value of the digit *N*?

3. How much greater is the product of the first four primes than the product of the first three composites?

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #1, October 2006 *Average score: 1.5 answers correct*

Answers

1. 102

2. 4

3. 18

1. Some people might be fooled by $3 \times 5 \times 7 = 105$, but $2 \times 3 \times 17 =$ **102** is the least three-digit number that is divisible by exactly three primes.

2. N clearly must be an even digit, so 0, 2, 4, 6, and 8 are possible. For $35N2N$ to be divisible by 3, the sum of the digits must be a multiple of three. So far we have $3 + 5 + N + 2 + N = 10 + 2N$. Let's use the possible values of N to evaluate this expression. If $N = 0$, then $10 + 2N = 10 + 0 = 10$, which is not divisible by 3. If $N = 2$, then $10 + 2N = 10 + 4 = 14$, which is not divisible by 3. If $N = 4$, then $10 + 2N = 10 + 8 = 18$, which *is* divisible by 3. Neither $N = 6$ nor $N = 8$ gives a multiple of 3, so 4 is our only candidate. Let's make sure that the number 35424 is divisible by all the other numbers listed. It is divisible by 4 since the last two digits 24 are divisible by 4. It is divisible by 6 since it passes the test for 2 and 3. It is divisible by 8 since the last three digits 424 are divisible by 8. It is divisible by 9 since the sum of the digits (18) is divisible by 9. Finally, it is divisible by 12 since it passes the test for 3 and 4. The digit N is indeed **4**.

Some Incorrect Answers Seen

1. 105, 110, 120, 1001

2. 9

3. 186, 6

3. The product of the first four primes is $2 \times 3 \times 5 \times 7 = 210$. The product of the first three composites is $4 \times 6 \times 8 = 192$. The difference is $210 - 192 =$ **18**.

Category 4
Arithmetic
Meet #1, October 2006

1. Evaluate the following expression according to the order of operations.

$$180 - 3 \times 7^2 + (20 - 34 \div 2)$$

2. The class scores on a math test are given in the stem-and-leaf plot at right. Find the value of the greatest of the mean, the median, or the mode.

Stem	Leaf
10	0
9	2 2 2 6
8	0 0 4 8 8 8 8 8
7	2 2 6 6 6
6	0 4 8 8
5	2

3. Jake was getting tired of dealing with small change at his hotdog stand. He decided to round all prices to the nearest quarter of a dollar. The old prices are given in the sign below. Last Thursday, the Roberts family ordered 2 hotdogs, 2 chilidogs, 1 tofudog, 5 chips, 1 water, 2 lemonades, and 2 milks. If they buy all the same items at the new rounded prices, how much more or less will they spend? Give your answer in dollars to the hundredth and write "more" or "less" after the amount.

Jake's Hotdogs

Hotdogs	$1.85
Chilidogs	$2.45
Tofudogs	$1.65
Chips	$0.95
Water	$0.90
Lemonade	$1.10
Milk	$0.80

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #1, October 2006

Average score: 1.6 answers correct

Answers

1. 36

2. 88

3. $0.05 more

1. The expression evaluates as follows:

$$180 - 3 \times 7^2 + (20 - 34 \div 2)$$
$$180 - 3 \times 49 + (20 - 17)$$
$$180 - 147 + 3$$
$$33 + 3$$
$$\mathbf{36}$$

2. The median (the middle score) is 80 and the mode (the most frequent) is 88. It seems very unlikely that the mean is greater than 88, so some students might be willing to answer **88** without actually calculating the mean. Indeed, the sum of the scores is 1840, and $1840 \div 23$ gives a mean of only 80.

3. To answer the question, we do not need to calculate how much they spent before or after the price changes. We will calculate the difference directly, using a negative sign for a decrease and a positive sign for an increase.

Hotdogs	$2 \times {}^{-}10¢$	$= {}^{-}20¢$
Chilidogs	$2 \times {}^{+}5¢$	$= {}^{+}10¢$
Tofudogs	$1 \times {}^{+}10¢$	$= {}^{+}10¢$
Chips	$5 \times {}^{+}5¢$	$= {}^{+}25¢$
Water	$1 \times {}^{+}10¢$	$= {}^{+}10¢$
Lemonade	$2 \times {}^{-}10¢$	$= {}^{-}20¢$
Milk	$2 \times {}^{-}5¢$	$= {}^{-}10¢$

The Roberts will spend 5 cents more, or **$0.05 more**, for the same items.

Category 5
Algebra
Meet #1, October 2006 *Average score: 1.9 answers correct*

1. Simplify the following expression.

$$3(4x+5)-5(8x+10)+7(4x+5)$$

2. Solve for x. (Note: Solve means to find the value of the variable that makes the equation true.)

$$7x+2(x+8)-13=606$$

3. Find the value of M that will make the equation below an identity. (Note: An identity is an equation that is true for all values of x.)

$$3x-(6-20x)+M=19x-7+4x+18$$

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #1, October 2006

Answers

1. 0

2. 67

3. 17

1. The expression simplifies as follows:

$$3(4x+5)-5(8x+10)+7(4x+5)$$
$$12x+15-40x-50+28x+35$$
$$40x-40x+50-50$$
$$\mathbf{0}$$

2. The equation can be solved as follows:

$$7x+2(x+8)-13=606$$
$$7x+2x+16-13=606$$
$$9x+3=606$$
$$9x=603$$
$$x=\mathbf{67}$$

3. First we simplify both sides of the equation.

$$3x-(6-20x)+M=19x-7+4x+18$$
$$3x-6+20x+M=23x+11$$
$$23x-6+M=23x+11$$

Clearly $23x$ will equal $23x$ for any value of x. If the equation is to be an identity, then $-6+M$ must equal 11. This is true when $M=\mathbf{17}$. Any other value of M will lead to an equation that has no solution.

Category 6
Team Questions
Meet #1, October 2006

1. The Blakes went to an ice cream place where a small cone costs $1.85, a medium cone costs $2.35, a large cone costs $2.75. There are 5 people in the Blake family and each person ordered one cone. If they spent a total ot $11.65 , how many medium cones did they buy?

2. Tia outlined a 3-by-3 block of dates on a calendar. The average of all nine numbers in the block is 15. What is the least number in the block of dates?

3. At the Euclid theater, $\frac{2}{5}$ of the seats are designated Orchestra seats, $\frac{1}{3}$ are Mezzanine, and $\frac{1}{4}$ are Balcony. There are 10 seats that are Obstructed View and 8 seats that are Wheelchair. How many seats are there at the Euclid theater?

4. What is the least four-digit number that is divisible by one of each of exactly four different prime numbers? That is, it is the product of four primes.

5. How many four-digit numbers meet the following criteria? All the digits are even. None of the digits is prime. None of the digits is a positive multiple of three. The mean (average) of the four digits in each number is 6. Note: Zero is an even number, but a four-digit number cannot start with zero.

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\sqrt{B+\frac{D-C}{A}+E-1}$$

Solutions to Category 6
Team Questions
Meet #1, October 2006 *Average score: 3.1 answers correct*

Answers

1. 3

2. 7

3. 1080

4. 1110

5. 9

6. 5

1. All the prices end in .05, so any combination of five ice-creams includes 25 cents. Let's subtract this 25 cents from \$11.65 to get \$11.40. Now we can think about the simpler prices of \$1.80, \$2.30, and \$2.70. Easier still is to use 18, 23, and 27 and try to get 114. If we start with five 18's, we get 90, which is too little. If we have five 23's, we get 115, which is close. Swapping a 23 for a 27, we go up to 119. Then swapping a 23 for an 18, we go down by 5 to 114. There must be one small cone, three medium cones, and one large cone. To verify we try \$1.85 + 3 × \$2.35 + \$2.75 = \$1.85 + \$7.05 + \$2.75 = \$11.65. So there are indeed three (**3**) mediums.

2. Since the average of the nine numbers is 15, the 15th is definitely the middle date. The nine numbers are 7, 8, 9, 14, 15, 16, 21, 22, and 23. The least number is **7**. The greatest number is **23**.

3. The sum of the fractions given is found as follows:

$$\frac{2}{5}+\frac{1}{3}+\frac{1}{4}=\frac{24}{60}+\frac{20}{60}+\frac{15}{60}=\frac{59}{60}$$

The 10 seats that are Obstructed View and the 8 seats that are Wheelchair must account for the other 60^{th} of the seats. If 18 is equal to $\frac{1}{60}$ of the seats, then there must be $18 \times 60 =$ **1080** seats at the Euclid theater.

4. It is more efficient to try to create this number by multiplying four different primes than it is to factor four-digit numbers looking for one with four different prime factors. We start by noting that $2 \times 3 \times 5 \times 7 = 210$ is not large enough and that $3 \times 5 \times 7 \times 11 = 1155$ is our first candidate. Now we wonder if we can find a smaller number that uses the factor of 2 and leaves out one of the other smaller primes. If we use 2, 3, and 5, which amounts to a product of 30, we will need to use 37 as our fourth prime to get a four-digit number. This is $2 \times 3 \times 5 \times 37 = 1110$, which is now our best candidate. If we use 2, 3, and 7, which gives us 42, we will need to multiply by 29 to make a four-digit number. This is $2 \times 3 \times 7 \times 29 = 1218$, which is greater than 1110. Other candidates are $2 \times 3 \times 11 \times 17 = 1122$, $2 \times 3 \times 13 \times 17 = 1326$, $2 \times 5 \times 7 \times 17 = 1190$, and $2 \times 5 \times 11 \times 13 = 1430$. The least is **1110**.

5. The nine (**9**) possible numbers are 4488, 4848, 4884, 8448, 8484, 8844, 8880, 8808, and 8088.
Note: The original meet question simply said "None of the digits is a multiple of three," which should eliminate 0 as a digit, so the answer to this question would then be 6 and not 9. The question was fixed for this archive copy.

6. Substituting the correct values into the expression, we get:

$$\begin{aligned}\sqrt{B+\frac{D-C}{A}+E-1} &= \sqrt{7+\frac{1110-1080}{3}+9-1} \\ &= \sqrt{7+\frac{30}{3}+9-1} \\ &= \sqrt{7+10+9-1} \\ &= \sqrt{25} \\ &= \mathbf{5}\end{aligned}$$

Note: In the original meet, because of question 5, an acceptable answer to this question was $\sqrt{22}$.